

# Taylor and Maclaurin Series

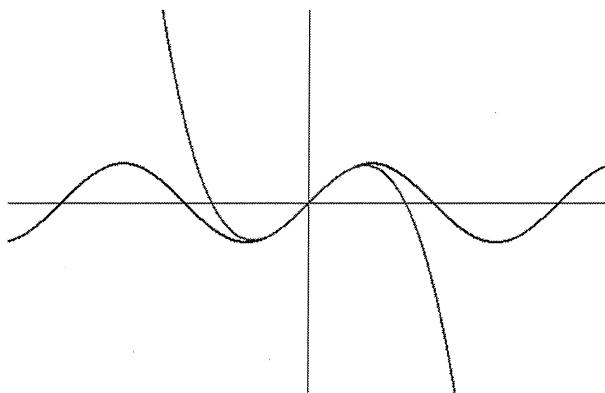
Anton 11.9

## Problem:

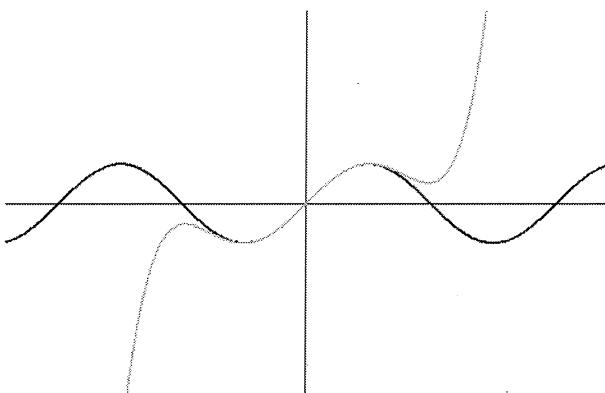
Given a function  $f$  and a point  $x = a$ , find a polynomial of specified degree that best approximates  $f$  in the “vicinity” of  $(a, f(a))$ .



Example: Could we approximate  $y = \sin(x)$  near  $x = 0$  with a cubic polynomial?



Example: Could we approximate  $y = \sin(x)$  near  $x = 0$  with a 5<sup>th</sup> degree polynomial?



Let's start by assuming that we will approximate  $f$  at  $x = 0$  with a polynomial of the form:

$$p_n(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n$$

Logic: This polynomial will be a good approximation of  $f$  if

$$p(0) = f(0)$$

$$p'(0) = f'(0)$$

$$p''(0) = f''(0) \quad \text{etc.}$$



Use this logic to find values of the constants:

$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots + c_nx^n$$

$$f(0) = p(0) = c_0 \rightarrow \boxed{c_0 = f(0)}$$

$$p'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots + nc_nx^{n-1}$$

$$f'(0) = p'(0) = c_1 \rightarrow \boxed{c_1 = f'(0)}$$



$$p''(x) = 2c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \dots + n(n-1)c_n x^{n-2}$$

$$f''(0) = p''(0) = 2c_2 \rightarrow c_2 = \frac{f''(0)}{2!}$$

$$p'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4 x + \dots + n(n-1)(n-2)c_n x^{n-3}$$

$$f'''(0) = p'''(0) = 3 \cdot 2c_3 \rightarrow c_3 = \frac{f'''(0)}{3!}$$

→

In general:  $c_n = \frac{\overbrace{f^{(n)}(0)}^{\text{n-th derivative}}}{n!}$

MacLaurin Polynomial for  $f$  about  $x = 0$ :

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

We can use a polynomial constructed in this way to approximate  $f$  near  $x = 0$ .

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Find the 3<sup>rd</sup> and 5<sup>th</sup> degree Maclaurin polynomials for  $y = \sin x$  at  $x = 0$ .

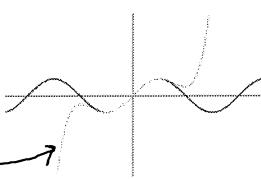
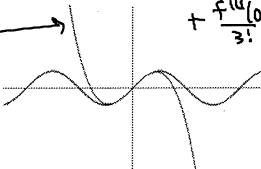
$$\left. \begin{array}{l} f = \sin x \\ f' = \cos x \\ f'' = -\sin x \\ f''' = -\cos x \\ f^{(4)} = \sin x \\ f^{(5)} = \cos x \end{array} \right|_{x=0} = \left. \begin{array}{l} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{array} \right.$$

$$P_3 = x - \frac{x^3}{3!}$$

$$P_5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$f \approx P_3 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$P_5 = P_3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$



Find the  $n$ th order Maclaurin polynomial

for  $y = e^x$ .

$$P_n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) = e^x \rightarrow f(0) = 1$$

$$f' = e^x \rightarrow f'(0) = 1$$

⋮

$$f^{(n)}(x) = e^x \rightarrow f^{(n)}(0) = 1$$

$$P_n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$



Taylor Polynomial for  $f$  about  $x = a$ :

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Maclaurin polynomial is just a Taylor polynomial with  $x = 0$ .



Find the third order Taylor polynomial for

$$y = \ln(x) \text{ at } x = 2. \quad P_3 = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$\left. \begin{array}{l} f = \ln x \rightarrow f(2) = \ln 2 \\ f' = \frac{1}{x} \rightarrow f'(2) = \frac{1}{2} \\ f'' = -\frac{1}{x^2} \rightarrow f''(2) = -\frac{1}{4} \\ f''' = \frac{2}{x^3} \rightarrow f'''(2) = \frac{2}{8} = \frac{1}{4} \end{array} \right\} \begin{aligned} P_3 &= \ln 2 + \frac{1}{2}(x-2) + \frac{-1/4}{2!}(x-2)^2 + \frac{1/4}{3!}(x-2)^3 \\ &= \ln 2 + \frac{(x-2)}{2} + \frac{-1}{4 \cdot 2!}(x-2)^2 + \frac{1}{4 \cdot 3!}(x-2)^3 \end{aligned}$$



Taylor Series for  $f$  about  $x = a$ :

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

MacLaurin Series for  $f$  about  $x = 0$ :

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$



Find the MacLaurin Series for each of the following:

$$y = \sin x \rightarrow f = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$y = e^x \rightarrow f = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$



Find the fourth order Taylor polynomial for  $f$  about  $x = 3$  and use it to approximate  $f(3.1)$ :

$$f(3) = 2, f'(3) = 1, f''(3) = -2, f'''(3) = -6, f^{(4)}(3) = 10$$

$$f = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \frac{f^{(4)}(3)}{4!}(x-3)^4$$

$$p_4 = 2 + (x-3) - \frac{2}{2!}(x-3)^2 - \frac{6}{3!}(x-3)^3 + \frac{10}{4!}(x-3)^4$$

$$f(3.1) \approx p_4(3.1) = 2 + .1 - (.1)^2 - (.1)^3 + \frac{5}{12} (.1)^4 = \underline{\quad 2.08904 \quad}$$

CALCULATOR



## Homework:

**Anton 11.9**  
**# 1,3,9,13,15,23,25,33,37**

